Lecture 6. Linear transformations in geometry

Prop The following functions are linear transformations on $\mathbb{R}^2$:

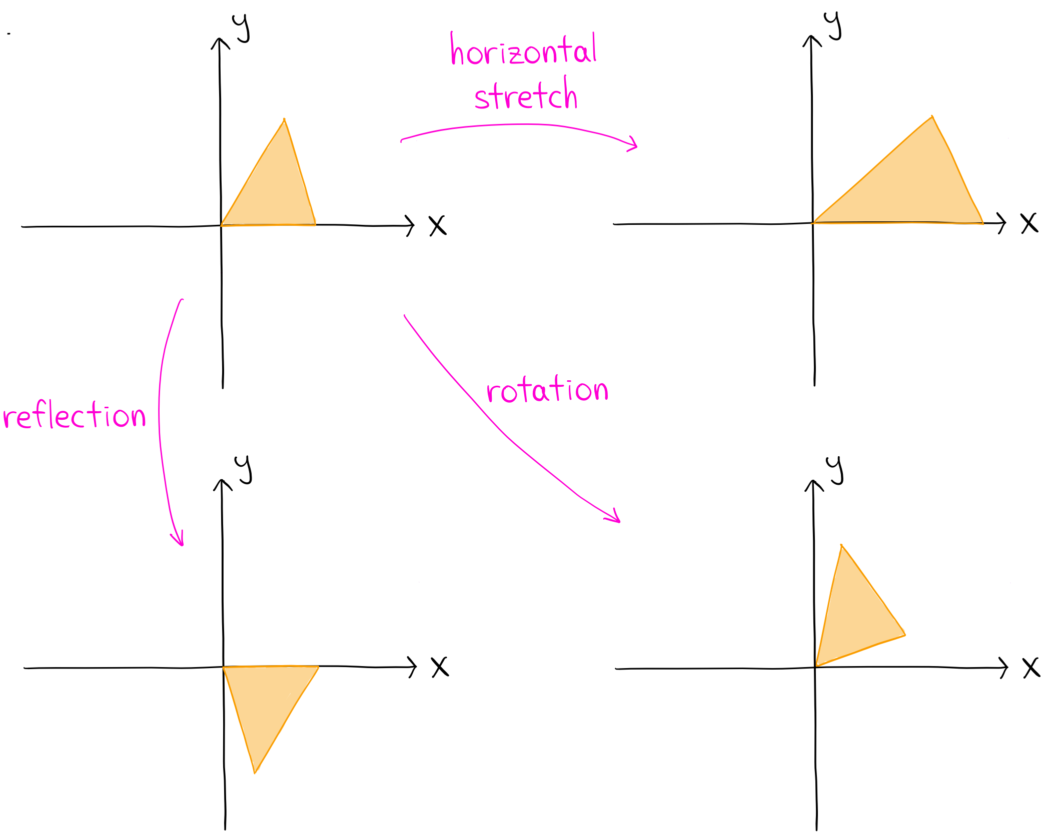
- multiplication by a fixed factor c,
- horizontal/vertical stretch by a fixed factor k,
- reflection through a fixed line $y = ax$,
- rotation about the origin through a fixed angle θ.

Note (1) By default, we take rotations to be counterclockwise.

(2) These transformations are useful for image processing.

We can stretch, reflect, rotate, or shear an image by applying a linear transformation to each point on the image.

e.g.



Ex For each linear transformation on $\mathbb{R}^2$, find the image of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

output

(1) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps $\vec{e}_1$ to $\vec{e}_1$ and $\vec{e}_2$ to $2\vec{e}_1 + \vec{e}_2$.

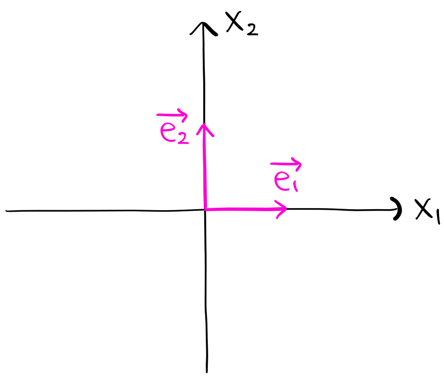
Sol $T_1(\vec{e}_1) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T_1(\vec{e}_2) = 2\vec{e}_1 + \vec{e}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

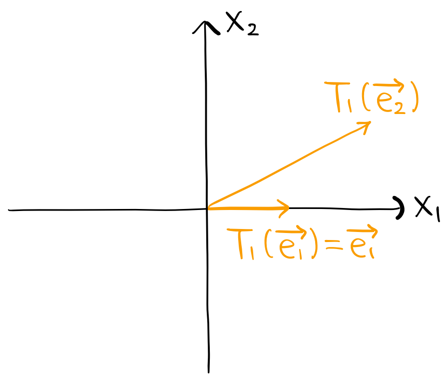
$\Rightarrow$ The standard matrix is

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow T_1(\vec{x}) = A_1\vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}}$$

Note T_1 is a shear transformation.

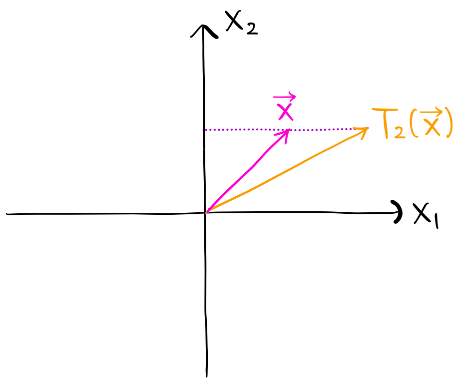




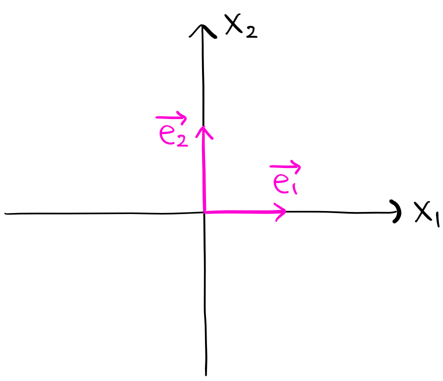
In fact, every shear transformation on $\mathbb{R}^2$ has standard matrix of the form

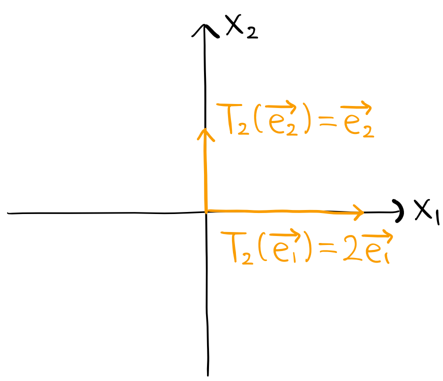
$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}.$$

(2) $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which stretches each vector horizontally by a factor 2.



Sol





$$T_2(\vec{e_1}) = 2\vec{e_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } T_2(\vec{e_2}) = \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

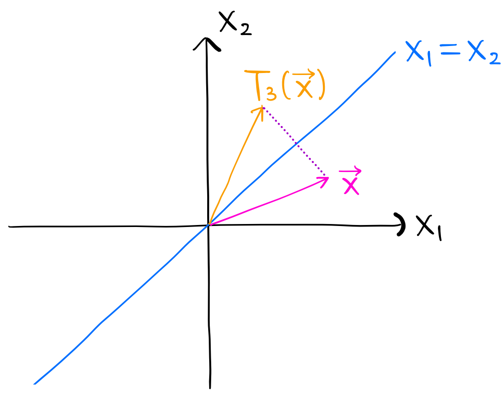
$\Rightarrow$ The standard matrix is

$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

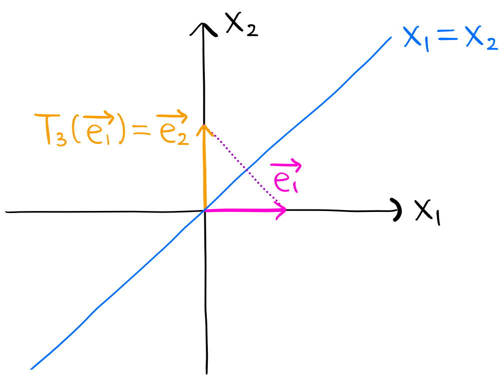
$$\Rightarrow T_2(\vec{x}) = A_2\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}}$$

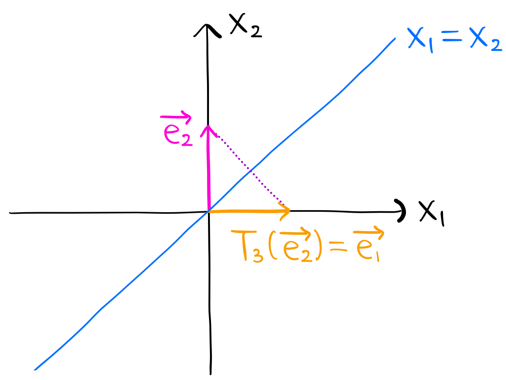
Note We can get the same answer by observing that T_2 multiplies the horizontal component by 2.

(3) $T_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which reflects each vector through the line $x_1 = x_2$.



Sol





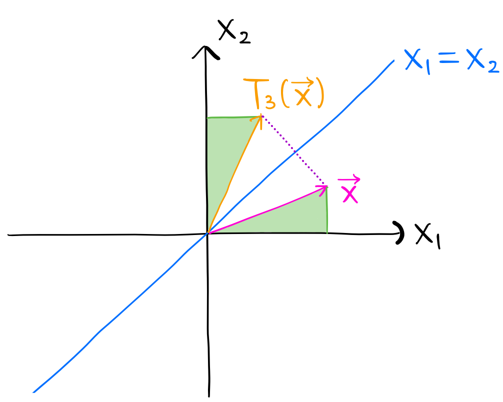
$$T_3(\vec{e_1}) = \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T_3(\vec{e_2}) = \vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\Rightarrow$ The standard matrix is

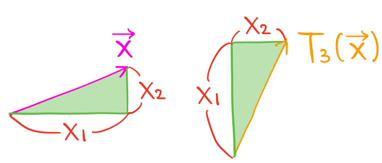
$$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\Rightarrow T_3(\vec{x}) = A_3\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}}$$

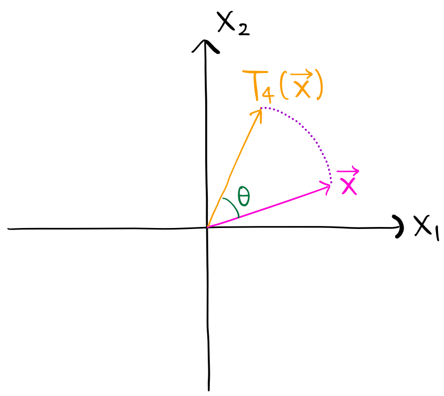
Note We can get the same answer by symmetry.



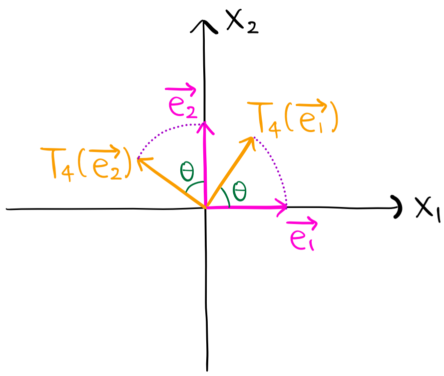
congruent triangles

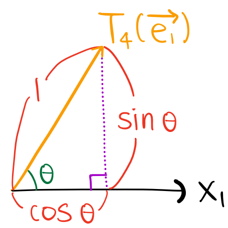


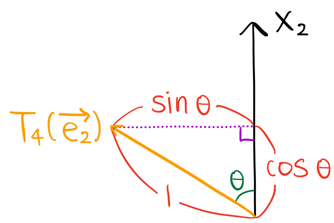
(4) $T_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which rotates each vector about the origin through a fixed angle θ.



Sol







$$T_4(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } T_4(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

$\Rightarrow$ The standard matrix is

$$\boxed{A_4 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}.$$

$$\Rightarrow T_4(\vec{x}) = A_4\vec{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}}$$